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# ON THE PROPAGATION OF SMOKE FROM FACTORY STACKS

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# ON THE PROPAGATION OF SMOKE FROM FACTORY STACKS

by

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I. A theoretical analysis of smoke propagation from factory stacks makes it possible to predict the degree of air pollution by smoke in an area with specific characteristics of the stacks and smoke, which is ejected into the atmosphere and under certain meteorological conditions. By knowing the latter, it is possible on the basis of a theoretical analysis to determine such characteristics of stacks and smoke with which the smoke concentration does not exceed the limits of certain established norms.

In an analysis of smoke diffusion it is important to take into consideration, generally speaking, that smoke particles are not suspended but possess a characteristic (proper) velocity. Frequently, as a result of overheating the smoke will possess an ascending velocity at a certain initial phase of its propagation. However, such an initial phase is of a short duration inasmuch as under the influence of the turbulent heat conductivity temperatures of the smoke particles and the surrounding air rapidly become equal. It is extremely difficult to calculate accurately the described effect because it would be necessary to solve simultaneously the diffusion equation and the equation for the free convection of smoke particles. However, even with a considerable overheating of smoke this effect may be calculated approximately by substituting the actual source of smoke with a fictitious - a somewhat elevated-source, (see [1]). Most frequently the effect is negligibly small due to the short duration of the mentioned first phase of smoke propagation, and at times it is totally absent. Therefore, vertical velocities caused by the overheating will not be taken into consideration. The descending characteristic velocities of smoke particles which are caused by the weight of particles are much more essential. For particles of a certain size the rate of their fall $w$ may be assumed as constant. Although the usual values of $w$ are extremely small, their influence may be significant, as long as it is realized throughout the entire time the particles are in motion with the exception, perhaps, of the mentioned initial phase. In addition, the existance of the rate of fall $w$ leads to certain qualitatively new effects.

Thus, if the smoke admixture is polydispersed, i.e. the smoke particles emerging from the chimney are of different sizes, then during the process of propagation of an admixture there occurs a re-distribution of particles according to the size - the heavier particles fall at a distance closer to the source than the lighter ones. Obviously, this fact would be impossible to describe, if the characteristic rate of fall of particles is ignored.

The presence of proper velocities $w$ leads also to another effect which at the first glance is paradoxical: with an increase of the strength of the source $Q$, i.e. the amount of smoke entering into the atmosphere

per unit of time, smoke concentration over a given distance from the source increases only up to a certain limit but with a further increment of Q, diminishes. This paradox is simple to explain, if it is taken into account that with the increment of Q the average size of smoke particles should also increase, and consequently also the mean value of $w$. The increment of Q alone leads to the increase of the concentration of admixture $q$ at any distance from the source, and the increase of $w$ - leads to the decrease of $q$. At small values of Q the direct influence of the increment of Q is more substantial, so that $q$ increases. However, with large Q values, the influence of the corresponding increase of the proper rate of fall $w$ is more substantial and therefore with the increase of Q the concentration of $q$ decreases.

The theory of the propagation of smoke admixture in the atmosphere and of analogous processes was examined in a number of works. Sutton [2] and M. E. Berliand [1] have not taken into account the characteristic rate of fall of smoke particles (see also [3]). Csanady [4] proposed to take into account approximately the individual velocity of $w$ by the means of substituting the axis of the source (i.e. with a horizontal straight line which is directed from the source with the wind), with a straight line inclined towards the axis downward at a certain angle. Evidently, such a method is no longer valid already owing to the fact that the line of the maximal values of $q$ differs substantially from the straight line in case $w \neq 0$.

M. I. IUdin [5], [6], as well as L. S. Gandin and A. S. Dubov [7] have examined a problem, which essentially is equivalent to the problem on two dimensional propagation of heavy admixtures, by ignoring the horizontal turbulent exchange or the same as the problem of propagation of admixture from a linear source. A similar problem, was examined by Fortak [8] but under a more general initial condition, with regard to the problem on the propagation of sand from the desert into the sea, whereupon contrary to the authors mentioned earlier, Fortak limited the problem to the simplest assumption on the constancy of the vertical turbulent coefficient exchange with height. The space problem on the propagation of heavy admixtures has been examined by Bosanquet and others [9], [10], who however, did not obtain a complete solution to this problem. This complete solution has been found recently by A. I. Denisov [11], who, by the way, established the error in one of the Bosanquet's formulas.

The present work is dedicated to the solution of the problem which is concerned with a series of more general relationships and which described more closely the behaviour of smoke under atmospheric conditions than the previously solved problems. In the analysis of the obtained solution attention is devoted to specific results caused by the effect of the proper rates of fall of smoke particles.

II. We shall solve the equation for the stationary turbulent diffusion of heavy particles

$$u\frac{\partial q}{\partial x} - w\frac{\partial q}{\partial z} = \frac{\partial}{\partial y}\left(k_y\frac{\partial q}{\partial y}\right) \quad \frac{\partial}{\partial z}\left(k_z\frac{\partial q}{\partial z}\right), \tag{1}$$

where x and y are the horizontal coordinates, with x coinciding with the wind direction and y perpendicular to the wind respectively, z - is the height, $k_y$ - is the coefficient of the horizontal turbulent diffusion, $k_z$ - is the coefficient of turbulent diffusion along the vertical, u - is the wind speed.

If q - is the concentration of particles with a certain characteristic size (radius) r, then the characteristic rate of fall, of w, may be assumed as constant in agreement with what was said previously. The wind velocity u and the coefficient $k_y$ shall be considered as power functions of height

$$u = u_1 z^m, \quad k_y = k_0 z^m \tag{2}$$

with the same exponent of the power m.

Such an approximation has been applied in a number of works (for example [1], [12], [13]) and proved to be successful. It should be noted that m is always a proper fraction. For practical purposes, as a rule $0.1 < m < 0.2$.

The coefficient $k_z$ will be assumed to increase linearly with height

$$k_z = k_1 z. \tag{3}$$

Let us solve the problem in the interval $0 \leqslant x < \infty$, $-\infty < y < +\infty$, $0 \leqslant z < \infty$. As the boundary conditions we shall assume the limitation of concentration at infinity

$$q|_{z=\infty} \neq \infty. \tag{4}$$

$$|q|_{y=\pm\infty} \neq \infty. \tag{5}$$

the given distribution of concentration in the "initial" plane

$$q|_{x=0} = F(y, z) \tag{6}$$

and let us also assume that the turbulent flux of admixture on the underlying surface is equal to zero

$$\left(k_z \frac{\partial q}{\partial z}\right)_{z=0} = 0. \tag{7}$$

Let us note that under condition (7) the total flux of admixture differs from zero owing to the sedimentation flux $wq|_{z=0}$, however, it is limited.

It may be indicated that for our problem condition (7) is equivalent to the condition of the limitation of concentration on the underlying surface.

$$|q|_{z=0} \neq \infty. \tag{8}$$

Under such formulations the problem on the propagation of heavy particles of smoke is closest to the problem of Bosanquet - Pearson, solved by Denisov but differs from it in the following relationships.

1. Instead of the "initial" condition for the type of source, condition (6) of the general type is used. It should be noted that the source of smoke in the form of an opening of a smokestack may, generally speaking, be considered as a point source, i.e. to assume that

$$F(y, z) = Q\delta(y)\delta(z-h), \tag{9}$$

where $h$ - is the height of the source, $\delta$ — is the delta-function*.

However, for a number of other practical problems, as for example for the problem mentioned above concerning the propagation of sand, it is necessary to use the condition in the initial plane of a more general form than (9); therefore, it would be useful to obtain a solution under the condition (6) especially as the method for the solution used in [11], under the condition of a source type cannot be applied directly in the case of condition (6). Of course, from the solution under the condition (6) it is easy to obtain a "source" solution, for which it is sufficient to determine $F(y, z)$ by equation (9) and to take advantage of the basic characteristic of the delta-function.

$$\int_a^b \Phi(x)\delta(x-c)\,dx = \Phi(c), \quad a < c < b. \tag{10}$$

2. The wind speed $u$ and the horizontal exchange coefficient $k_y$ are not considered as constant but increase with height according to (2). It is clear that this gives the best approximation of the actual atmospheric conditions.

3. The coefficient $k_y$ is assumed to be independent of the horizontal coordinate $x$, while in works [9] - [11] it is assumed that it is proportional to $x$. The last assumption used in a number of works by the English authors, is based on the fact that the intensity of the turbulent mixing of the admixture should depend on the characteristic dimensions of the admixture "cloud" increasing with their growth; since the admixture "cloud" expands in the direction $x$ then they assume that $k_y$ increases with an increment of $x$. Such an assumption is not very convincing owing to the following reasons.

Firstly, if it would have been possible to agree with the arguments presented above, then they would pertain not only to the coefficient $k_y$ but equally to the coefficient $k_z$, so long as the admixture "cloud" expands with the increase of $x$ along the vertical also. In addition to this, these coefficients should be considered as independent of the $x$ coordinates but dependent on the distribution of the diffusing admixture $q(x, y, z)$, i.e. a non-linear problem should be solved. Secondly, it is not quite clear in what manner these arguments may be used in case of the initial distribution of a general type (6), instead of a source type.

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* $\delta$-function: $\delta(t) = \lim\limits_{h\to 0} f(t,h)$, where $f(t,h) = \frac{1}{h}$ at $0 < t < h$, and $f(t,h) = 0$ outside of this interval. (Translator)

Finally, thirdly, numerous theoretical investigations exist in which the coefficient of the horizontal mixing was assumed as independent of the horizontal coordinates (for example [1], [3], [13]). A good agreement between the theoretical results and the observational data shows that the coefficients of turbulent diffusion may be considered independent of the horizontal coordinates.

III. By substititing (2) and (3) into (1) we shall obtain

$$u_1 \frac{\partial q}{\partial x} - wz^{-m} \frac{\partial q}{\partial z} = k_1 z^{-m} \frac{\partial}{\partial z}\left(z \frac{\partial q}{\partial z}\right) + k_0 \frac{\partial^2 q}{\partial y^2}. \tag{11}$$

Let us assume at first that the distribition of q in the initial plane is symmetrical with respect to the axis z, i.e., that function F (y, z) - is an even function according to y . Then by multiplying equation (11) by

$$Y(y;\lambda) = \cos \lambda y \tag{12}$$

and assuming that

$$\int_{-\infty}^{\infty} q(x, y, z)\, Y(y;\lambda)\, dy = \Psi(x, z; \lambda) \tag{13}$$

and

$$\int_{-\infty}^{\infty} F(y, z)\, Y(y;\lambda)\, dy = f(z;\lambda). \tag{14}$$

We obtain according to (12) and (13)

$$q(x, y, z) = \frac{1}{\pi} \int_0^{\infty} \Psi(x, z; \lambda)\, Y(y;\lambda)\, d\lambda. \tag{15}$$

By integrating equation (11), which is multiplied by Y , from $-\infty$ to $\infty$ with respect to the variable y , we shall obtain, by using condition (5),

$$u_1 \frac{\partial \Psi}{\partial x} - wz^{-m} \frac{\partial \Psi}{\partial z} = k_1 z^{-m} \frac{\partial}{\partial z}\left(z \frac{\partial \Psi}{\partial z}\right) - k_0 \int_{-\infty}^{\infty} \frac{\partial^2 q}{\partial y^2}\, Y\, dy$$

or after integrating twice by parts in the last term

$$u_1 \frac{\partial \Psi}{\partial x} - wz^{-m} \frac{\partial \Psi}{\partial z} = k_1 z^{-m} \frac{\partial}{\partial z}\left(z \frac{\partial \Psi}{\partial z}\right) - k_0 \lambda^2 \Psi. \tag{16}$$

Further let us perform an operational transformation according to x . Since according to (6), (13) and (14)

$$\Psi |_{x=0} = f.$$

then the transformation of equation (16) gives

$$u_1 p (\overline{\Psi} - f) - wz^{-m} \frac{d\overline{\Psi}}{dz} = k_1 z^{-m} \frac{d}{dz}\left(z \frac{d\overline{\Psi}}{dz}\right) - k_0 \lambda^2 \overline{\Psi}.$$

or

$$\frac{d^2\overline{\Psi}}{dz^2} + \left(1 + \frac{w}{k_1}\right) \frac{1}{z} \frac{d\overline{\Psi}}{dz} - \frac{k_0\lambda^2 + u_1 p}{k_1} z^{m-1} \overline{\Psi} = \frac{u_1 p}{k_1} z^{m-1} f. \tag{17}$$

where $\overline{\Psi}$ — is the transform of function $\Psi$.

The general solution of equation (17) may be written out in the form:

$$\bar{\Psi} = \frac{2u_1 p}{(1+m)k_1}\left\{\left[\int_z^{\infty} f(\zeta;\lambda)\,\zeta^{m+\frac{w}{2k_1}} z^{-\frac{w}{2k_1}} \cdot K_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,\zeta^{\frac{1+m}{2}}\right)d\zeta + \right.\right.$$

$$\left. + A\right] I_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,z^{\frac{1+m}{2}}\right) +$$

$$+\left[\int_0^{z} f(\zeta;\lambda)\,\zeta^{m+\frac{w}{2k_1}} z^{-\frac{w}{2k_1}}\, I_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,\zeta^{\frac{1+m}{2}}\right)d\zeta +\right.$$

$$\left.\left. + B\right] K_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,z^{\frac{1+m}{2}}\right)\right\},$$

where A and B - are constants of integration. For determining them we shall use the boundary conditions (4) and (8), which lead to the conditions for $\bar{\Psi}$,

$$|\bar{\Psi}|_{z=\infty} \neq \infty, \quad |\bar{\Psi}|_{z=0} \neq \infty.$$

As a result of the first of these conditions A = 0, and owing to the second B = 0, therefore the solution for $\bar{\Psi}$ obtains the form

$$\bar{\Psi} = \frac{2u_1 p z^{-\frac{w}{2k_1}}}{(1+m)k_1}\left[\int_z^{\infty} f(\zeta;\lambda)\,\zeta^{m+\frac{w}{2k_1}} K_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,\zeta^{\frac{1+m}{2}}\right)\times\right.$$

$$\times\, d\zeta \cdot I_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,z^{\frac{1+m}{2}}\right) + \int_0^{z} f(\zeta;\lambda)\,\zeta^{m+\frac{w}{2k_1}}\times$$

$$\left.\times\, I_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,\zeta^{\frac{1+m}{2}}\right)d\zeta\, K_{\frac{w}{(1+m)k_1}}\left(\frac{2}{1+m}\sqrt{\frac{k_0\lambda^2+u_1 p}{k_1}}\,z^{\frac{1+m}{2}}\right)\right]. \quad (18)$$

Owing to (18), $\left(k_1 z \frac{\partial \bar{\Psi}}{\partial z}\right)_{z=0} = 0$ . Therefore, the solution also satisfies condition (7).

In order to obtain the original $\Psi(x, z; \lambda)$ from the transform $\bar{\Psi}(z; p, \lambda)$ let us use the well known formula [see for instance [14], formula (9. 140)].

$$pK_\nu[(\sqrt{\alpha}+\sqrt{\beta})\sqrt{p+\gamma}]\,I_\nu[(\sqrt{\alpha}-\sqrt{\beta})\sqrt{p+\gamma}] \div \frac{1}{2t} e^{-\gamma t-\frac{\alpha+\beta}{2t}} I_\nu\left(\frac{\alpha-\beta}{2t}\right),$$

where t in this case is the coordinate x.

Then by combining the integrals we shall obtain

$$\Psi = \frac{u_1}{(1+m)\,k_1 x}\, e^{-\frac{k_0\lambda^2 x}{u_1}} \int_0^\infty f(\zeta;\ \lambda)\,\zeta^m \left(\frac{\zeta}{z}\right)^{\frac{w}{2k_1}} e^{-\frac{u_1(\zeta^{1+m}+z^{1+m})}{(1+m)^2 k_1 x}} \times$$

$$\times I_{\frac{w}{(1+m)k_1}}\left[\frac{2u_1 (z\zeta)^{\frac{1+m}{2}}}{(1+m)^2 k_1 x}\right] d\zeta. \tag{19}$$

Returning to q , we obtain, according to (12), (14), (15) and (19)

$$q = \frac{u_1}{(1+m)\,k_1 \pi x} \int_0^\infty \int_{-\infty}^{\infty}\int_0^\infty F(\eta,\ \zeta)\cos\lambda y\cos\lambda\eta\, e^{-\frac{k_0\lambda^2 x}{u_1}} \zeta^m \left(\frac{\zeta}{z}\right)^{\frac{w}{2k_1}} \times$$

$$\times e^{-\frac{u_1(\zeta^{1+m}+z^{1+m})}{(1+m)^2 k_1 x}}\, I_{\frac{w}{(1+m)k_1}}\left[\frac{2u_1 (z\zeta)^{\frac{1+m}{2}}}{(1+m)^2 k_1 x}\right] d\lambda\, d\eta\, d\zeta$$

or integrating with respect to $\lambda$, finally

$$q = \frac{1}{4(1+m)\sqrt{\pi k_0}\,k_1}\left(\frac{u_1}{x}\right)^{\frac{3}{2}} \int_0^\infty \int_{-\infty}^{\infty} F(\eta,\ \zeta)\left[e^{-\frac{u_1(y-\eta)^2}{4k_0 x}} + \right.$$

$$\left. + e^{-\frac{u_1(y+\eta)^2}{4k_0 x}}\right]\zeta^m \left(\frac{\zeta}{z}\right)^{\frac{w}{2k_1}} e^{-\frac{u_1(\zeta^{1+m}+z^{1+m})}{(1+m)^2 k_1 x}}\, I_{\frac{w}{(1+m)k_1}}\left\{\frac{2u_1 (z\zeta)^{\frac{1+m}{2}}}{(1+m)^2 k_1 x}\right\} d\eta\, d\zeta. \tag{20}$$

The problem would be solved analogously when F (y, z) is the odd function of the coordinate y . Then $\sin\lambda y$, should be taken as a factor for Y and according to this the final formula will differ from formula (20) only by the fact that instead of the sum of the exponent the difference will appear in square brackets. Finally, in general case it is simple to obtain the solution, by taking the function F (y, z) in the form of the sum of the even and the odd function

$$F(y,\ z) = \frac{1}{2}\,[F(y,\ z) + F(-y,\ z)] + \frac{1}{2}\,[F(y,\ z) - F(-y,\ z)].$$

Then we obtain

$$q = \frac{1}{4(1+m)\sqrt{\pi k_0}\,k_1}\left(\frac{u_1}{x}\right) \int_0^\infty \int_{-\infty}^{\infty}\left[F(\eta,\ \zeta)\, e^{-\frac{u_1(y-\eta)^2}{4k_0 x}} + \right.$$

$$\left. + F(-\eta,\ \zeta)\, e^{-\frac{u_1(y+\eta)^2}{4k_0 x}}\right]\zeta^m \left(\frac{\zeta}{z}\right)^{\frac{w}{2k_1}} \times$$

$$\times e^{-\frac{u_1(\zeta^{1+m}+z^{1+m})}{(1+m)^2 k_1 x}}\, I_{\frac{w}{(1+m)k_1}}\left\{\frac{2u_1 (z\zeta)^{\frac{1+m}{2}}}{(1+m)^2 k_1 x}\right\} d\eta\, d\zeta. \tag{21}$$

Formula (21) gives a complete solution of the stated problem. In particular, for the point source (9) we obtain from (21) or (20) on the basis of (10)

$$q = \frac{\left(\frac{u_1}{x}\right)^{\frac{3}{2}} Qh^m}{2(1+m)\sqrt{\pi k_1} k_1} \left(\frac{h}{z}\right)^{\frac{w}{2k_1}} e^{-\frac{u_1 y^2}{4k_0 x} - \frac{u_1 (h^{1+m} + z^{1+m})}{(1+m)^2 k_1 x}} I_{\frac{w}{(1+m)k_1}} \left\{\frac{2u_1 (hz)^{\frac{1+m}{2}}}{(1+m)^2 k_1 x}\right\}. \qquad (22)$$

From (22) are obtained simple expressions for the distribution of the axial concentration $q_h$ in case the source is elevated $(y=0,\ z=h)$

$$q_h = \frac{\left(\frac{u_1}{x}\right)^{\frac{3}{2}} Qh^m}{2(1+m)\sqrt{\pi k_0} k_1} e^{-\frac{2u_1 h^{1+m}}{(1+m)^2 k_1 x}} I_{\frac{w}{(1+m)k_1}} \left\{\frac{2u_1 h^{1+m}}{(1+m)^2 k_1 x}\right\} \qquad (23)$$

and for the distribution of the axial concentration $q_0$ in case the source is at ground level $(y=z=0)$

$$q_0 = \frac{\left(\frac{u_1}{x}\right)^{\frac{3}{2}} Qh^{m+\frac{w}{k_1}}}{2(1+m)\Gamma\left\{1+\frac{w}{(1+m)k_1}\right\}\sqrt{\pi k_0} k_1} \left[\frac{u_1}{(1+m)^2 k_1 x}\right]^{\frac{w}{(1+m)k_1}} e^{-\frac{u_1 h^{1+m}}{(1+m)^2 k_1 x}}. \qquad (24)$$

IV In analyzing the solution it is convenient to change to dimensionless variables by means of relations

$$\sqrt{\frac{u_1}{k_0 x}}\,\frac{y}{2} = \eta, \quad \frac{w}{(1+m)k_1} = p, \quad \frac{2}{(1+m)}\sqrt{\frac{\pi k_0}{k_1}}\, h^{1+m}\frac{q}{Q} = \sigma,$$

$$\frac{1}{1+m}\sqrt{\frac{u_1}{k_1 x}}\, z^{1+m} = \zeta, \quad \frac{1}{1+m}\sqrt{\frac{u_1}{k_1 x}}\, h^{1+m} = H. \qquad (25)$$

Then instead of formula (22) we obtain

$$\sigma(\eta, \zeta; H) = H^3 \left(\frac{H}{\zeta}\right)^p e^{-\eta^2 - H^2 - \zeta^2} I_p(2H\zeta) \qquad (26)$$

and correspondingly instead of formula (23) and (24)

$$\sigma(0, H; H) = H^3 e^{-2H^2} I_p(2H^2) \qquad (27)$$

and

$$\sigma(0, 0; H) = \frac{1}{\Gamma(1+p)} H^{3+2p} e^{-H^2}. \qquad (28)$$

Formula (27) shows that the axial concentration for an elevated source decreases monotonically with an increase in the distance from the axis of the source, i.e., with the decrease of H . Along all the other horizontal straight lines in the plane y=0 the admixture concentration at first increases with distance from the point x=0 , and only after reaching a certain maximum begins to decrease. At the axis of the ground source in particular, the maximum value σ, according to (28), is reached at the point

$$H = H_m = \sqrt{p + \frac{3}{2}} \qquad (29)$$

and is equal to

$$\sigma_m = \sigma(0,\ 0;\ H_m) = \frac{1}{\Gamma(1+p)}\left(\frac{p+\frac{3}{2}}{e}\right)^{p+\frac{3}{2}}. \tag{30}$$

It is interesting to note that the distance

$$x_m = \frac{1}{(1+m)^2\left[\frac{w}{(1+m)k_1}+\frac{3}{2}\right]}\frac{u_1}{k_1}h^{1+m}, \tag{31}$$

at which, according to (29) and (25) the maximum concentration is reached, is independent of the horizontal diffusion coefficient $k_0$, and the maximum concentration, being equal, according to (30) and (25)

$$q_m = \frac{(1+m)^2}{2}\frac{Q\sqrt{k_1}}{\sqrt{\pi k_0 h^{3+m}}}\frac{1}{\Gamma\left\{1+\frac{w}{(1+m)k_1}\right\}}\left[\frac{\frac{w}{(1+m)k_1}+\frac{3}{2}}{e}\right]^{\frac{w}{(1+m)k_1}+\frac{3}{2}}, \tag{32}$$

is independent of the wind speed $u_1$.

Both of these conclusions are clear from simple physical considerations. Indeed, the maximum concentration should be dependent only on time $\frac{x}{u_1}$, in which particles are transported by the wind from the source to the point of the abscissa $x$, and this time, apparently, is not dependent on $u_1$. Regarding the coefficient $k_0$, its value should effect the rate of maximum concentration, but at the point location where such a concentration is reached.

Let us note that the presence of a point (source) with a maximum concentration at the underlying surface is not characteristic for the diffusion of heavy particles only. In the case of weighted particles the general character of the distribution of $q$ on a straight line $y=z=0$ is the same. In this respect the influence of the proper weight of particles of the admixture is of a quantitative nature only. Namely, as may be seen from (31) and (32), with the increase of the deposition rate of particles, the point at which the maximum concentration is reached, approaches the source but the value of the maximum concentration increases.

In general, the spacial distribution of a concentration does not undergo qualititative changes under the influence of the weight of admixture particles. Therefore, we shall not describe the distribution of $q$ any further but we shall turn to the analysis of specific effects caused by the influence of the proper rates of fall of particles.

Let us assume that the admixture is polydispersed and the distribution curve for the number of particles emitted by the source, according to the sizes of $r$ is described by the function $f(r)$. For simplification

we shall assume the particles to be spherical and r will be considered as the radius of the particles. These considerations may easily be generalized in the case of particles which are arbitrary in shape, providing there is no relationship between the size and the shape of particles.

If $N_0$ is the total number of admixture particles emitted by the source per unit of time, then according to what has been said,

$$N_0 f(r)\, dr$$

is the number of particles (among the total number $N_0$) with a radius between r and r + dr. Then the strength of the source Q is equal, obviously, to

$$Q = \int \frac{4}{3} \pi r^3 \rho N_0 f(r)\, dr,$$

where $\rho$ — is the density of admixture substance and the integration is made throughout the entire spectrum of the size of particles.

Let us assume that the size of every particle does not change during the diffusion process of the admixture and that no interaction occurs between the particles. This assumption may lead to certain errors only at the immediate proximity of the source, but at a distance from it, where the concentration, q, is small, it is very close to reality. Following this assumption it is possible to examine the behaviour of particles of every fixed size independently.

Let us substitute in (28)

$$\sigma(0,\ 0,\ H) = \frac{1}{Q} s,$$

in order to single out the dependency of the concentration on Q.

Then formula (28) for the concentration of particles of a given size at the ground level axis of the source will be written in the form

$$s = \frac{Q_r}{\Gamma(1+p_r)} H^{3+2p_r} e^{-H^n}, \qquad (33)$$

where, in addition to

$$Q_r = \frac{4}{3} \pi r^3 \rho N_0 f(r), \qquad (34)$$

also only the parameter $p = p_r$ depends on the radius r of the particles, so long as it is related according to (25) to the deposition rate of particles. Namely, owing to the smallness of particle sizes it is natural to assume their deposition rate proportional to the square of the radius

$$w = w_0 \left(\frac{r}{r_0}\right)^2,$$

then

$$p = \frac{w_0}{(1+m) k_1 r_0^2} r^2. \qquad (35)$$

By having a concrete distribution curve $f(r)$ of particles which are emitted by the source, according to sizes and knowing the parameter $\frac{w_0}{r_0^2}$[1], it is possible on the basis of (33), (34) and (35) to find the distribution of the concentration of particles of given sizes at the underlying surface. After this, integrating with respect to $r$, it is also possible to find the distribution of total concentration. In addition to this, it is essential to keep in mind that the distribution of total concentration will vary from the distribution of the corresponding monodispersed admixture, which is obtained by substituting the sizes of all particles with the average sizes. In particular, at the earth's surface the total concentration of polydispersed admixture will be greater than the corresponding concentration of the monodispersed admixture.

It is even more essential to take into account the re-distribution of particles according to the size in the propagation process of admixture. By considering the concentration at the underlying surface, it is natural to introduce the coefficient

$$\chi(r) = \frac{H^{2p_r}}{\Gamma(1+p_r)}, \tag{36}$$

which describes, according to (33) the distribution of $s$ according to size if $\frac{dQ_r}{dr} = 0$, i.e., in the event, the strength of the source for various size particles is the same. In other words, the coefficient $\chi(r)$ describes the influence of the deposition of particles on the distribution of their mass according to their sizes over the underlying surface.

The behaviour of function $\chi(r)$ differs depending on $H$. Namely, for small $H$ values (greater distances of $x$) $\chi(r)$ has a monotonic decrease with the increase of $r$, but for large $H$ values (small $x$ values) $\chi(r)$ increases to a certain maximum and only after that it decreases. It is simple to describe this effect quantitatively, if we keep in mind that $p_r << 1$ (due to the smallness of the parameter $\frac{w_0}{[(1+m)k_1]}$). The condition of the maximum of $\chi(r)$ gives

$$2\ln H - \psi(1+p) = 0,$$

where $\psi$ — is the logarithmic derivative of a gamma-function.

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[1] If, for example, the deposition of particles takes place according to Stokes law, then
$$\frac{w_0}{r_0^2} = \frac{2\rho g}{9\mu}$$

By the expansion of the $\psi$ function into a series and by confining to two terms

$$\psi(1+p) = -1 + \frac{\pi^2 p}{6},$$

we obtain the value of p , corresponding to the maximum of $\chi(r)$.

$$p_m = \frac{12}{\pi^2}\left(\ln H + \frac{1}{2}\right). \tag{37}$$

Since always $p > 0$, then, according to (37) the maximum of $\chi(r)$ occurs only if

$$\ln H > -\frac{1}{2},$$

i.e., in agreement with (25), if

$$x < x_n = \frac{e}{(1+m)^2} \frac{u_1}{k_1} h^{1-m}. \tag{38}$$

A physical explanation of this fact consists in the following. Generally speaking, the larger are the particles then to a greater degree is their concentration decreased at a given distance from the source, because a large portion of such particles, owing to deposition, fall out at closer distances. But at short distances (38), the concentration of very fine particles is small owing to the fact that as a result of the slow deposition rate they do not quite reach the underlying surface. At moderate distances with all conditions being equal, there are mostly particles of intermediate sizes which satisfy formula (37).

It is interesting to note that the upper limit $x_n$ of the distances, where the curve $\chi(r)$ reaches the maximum, is very simply associated with the abscissa $x_m$ , for which the concentration of particles of a certain given size is maximal. According to (31) and (38),

$$\frac{x_n}{x_m} = e\left[\frac{w}{(1+m)k_1} + \frac{3}{2}\right] \approx \frac{3}{2}e \approx 4{,}38.$$

Finally, let us examine the problem concerning the relationship between the concentration of particles and the strength of the source. It is obvious that if the process of the propagation of admixture is independent of the strength of the source, then at each point the concentration is proportional to the strength of the source and increases without limit with the increase of the strength of the source. Actually, however, this is more complicated. Namely, with the change of the strength of the source, the distribution curve of the size particles being emitted by it also changes, so that with an increase in the strength of the source, the average particle sizes increase also. Therefore, the particles settle more intensely and it can be assumed that at a certain far distance from the source, as a result, the concentration with the increase of the strength of the source does not grow but diminishes.

Let us examine the simplest physical scheme of such an effect and namely let us assume that

a) the admixture is monodispersed and

b) the number of particles which is emitted by the source per unit of time is independent of the strength of the source.

Then it is obvious that

$$Q = Q_0 \left(\frac{r}{r_0}\right)^3. \tag{39}$$

where $Q_0$ - is the value of Q at $r=r_0$.

Owing to (35) and (39)

$$Q = Q_0 \left(\frac{p}{p_0}\right)^{\frac{3}{2}} \quad (p_0 = p_{r=r_0}). \tag{40}$$

By substituting (40) into (33) we obtain

$$s = \frac{Q_0}{p_0^{\frac{3}{2}}} \frac{H^{3+2p}}{\Gamma(1+p)} p^{\frac{3}{2}} e^{-Hp}. \tag{41}$$

The condition of the maximum of s with respect to p gives, according to (41),

$$3 + 4p \ln H - 2p\psi(1+p) = 0,$$

from which by confining to the first term of the expansion of $\psi(1+p)$ into a series

$$p = -\frac{3}{4} \frac{1}{\ln H + \frac{1}{2}}. \tag{42}$$

From (42) it is evident that s as a function of p (or the same as a function of Q ) has a maximum only if

$$\ln H < -\frac{1}{2},$$

i.e., if

$$x > x_n = \frac{e}{(1+m)^2} \frac{u_1}{k_1} h^{1+m}.$$

The lower limit $x_n$ of these x values coincides with the upper limit of formula (38). Thus, for all sufficiently large distances of $x > x_n$ the concentration with an increase in the strength of the source does not increase infinitely but reaches a certain maximum with the Q values, which is obtained by substituting (42) into (40), and with a further increase Q decreases. The values of p and correspondingly of Q at which the indicated maximum is reached, essentially depend on the distance x from the source, and namely

$$p = \frac{3}{2\ln\frac{x}{x_0}},$$

$$Q = Q_0\left[\frac{3}{2p_0\ln\frac{x}{x_0}}\right]^{\frac{3}{2}}.$$

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